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Convex hulls of a curve in control theory

A. N. Kurbatskiĭ

Abstract. A classification is obtained for typical singularities of the local transitivity sets of control systems on three-dimensional manifolds with nonconvex indicatrices that are closed smooth spatial curves.

Bibliography: 8 titles.

Keywords: transitivity set, singularity, convex hull.

§ 1. Introduction

We consider a control system on a three-dimensional manifold M whose tangent space T_mM at each point m is equipped with the set I_m of admissible velocities (indicatrix) which depends smoothly on m. We assume that I_m is a smooth closed spatial curve. The case when I_m is a smooth closed surface was considered in [1].

The set of local transitivity, consisting of those points m for which the zero velocity $O \in T_mM$ belongs to the interior of the convex hull of I_m, is of interest in control theory. In particular, for each pair of neighbouring points of the set of local transitivity, there is a sufficiently short admissible curve connecting these points. Recall that an absolutely continuous parametrized curve $\gamma(t)$ is referred to as *admissible* if the derivative $\dot{\gamma}(t)$ at almost every point belongs to the indicatrix of this point. The boundary Σ of the set of local transitivity consists of those points m for which O belongs to the boundary $H(I_m)$ of the convex hull of the indicatrix (see [2]).

In this work we classify all typical local singularities of Σ up to diffeomorphism (Theorem 2). All these singularities are germs of graphs of Lipschitz continuous functions and functions of class C^1.

The classification problem for singularities of the boundary Σ was proposed by A. A. Davydov. The author is grateful to him for useful discussions.

§ 2. Main results

Treating a point m of the manifold M^n as a parameter and fixing a trivialization of the tangent bundle of M, we reduce the initial problem to the study of a family of curves $r_m(t)$, $t \in S^1$, embedded in $\mathbb{R}^n$ and depending on the parameter m, and to the classification of singularities of the set Σ of those parameter values for which

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$0 \in \mathbb{R}^n$ belongs to the boundary $H(r_m)$ of the convex hull of the corresponding curve.

We start with a simple case of the control system

$$\dot{x} = f(x, u), \qquad x \in M^2, \quad u \in S^1,$$

on a two-dimensional manifold M whose tangent plane at each point is equipped with the set of admissible velocities I_m, and assume that this set is a closed smooth curve.

As we shall see below, the list of typical singularities of Σ includes the list of typical singularities of the convex hulls of individual smooth curves.

In the case of a generic curve in the plane the list is as follows:

- the germ of a convex curve or a line;
- the germ of a curve which can be taken by a diffeomorphism of the plane to the germ at the origin of the graph of function $y = f(x)$, where

$$f(x) = \begin{cases} 0 & \text{for } x \leqslant 0, \\ x^2 & \text{for } x > 0. \end{cases}$$

For a typical family of curves depending on two parameters, the boundary Σ of the set of local transitivity can have only one more new singularity.

Theorem 1. *For a generic family of curves I_m in $\mathbb{R}^2$ depending on a parameter $m = (x, y) \in \mathbb{R}^2$, a local singularity of the boundary Σ of the set of local transitivity is either a typical singularity of the convex hull listed above, or the germ at the origin of the graph of function $y = -|x|$ (up to a diffeomorphism of the space of parameters).*

The proof is given in the next section (after the proof of main Theorem 2).

We now turn to the main case of a three-dimensional manifold M. The convex hull of a typical smooth spatial curve (of class C^∞) is not smooth. The following list was obtained in [3] and [4]:

The list of typical singularities of the convex hull of a curve in $\mathbb{R}^3$:

(1) the germ of a smooth surface which is either developable (Fig. 1) or planar (Fig. 2, point C);

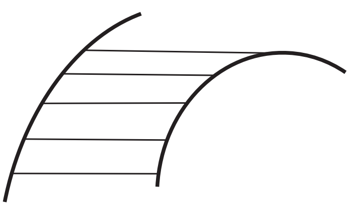

Figure 1

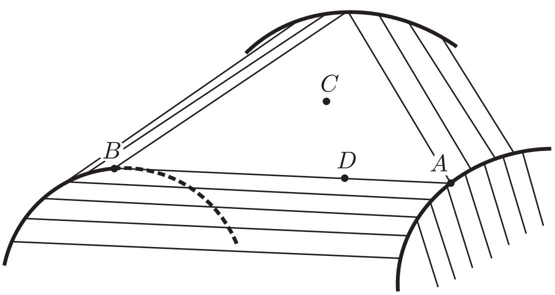

Figure 2

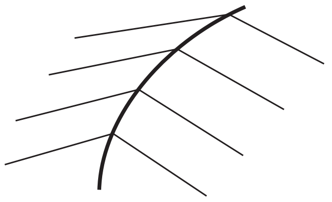

Figure 3

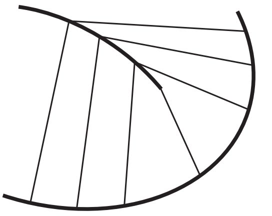

Figure 4

(2) the germ at the origin of the graph of Lipschitz continuous function $z = f(x, y)$, where

$$f(x, y) = -|x|$$

(this germ appears at a typical point of the initial curve itself; see Fig. 3);

(3) the germ of class C^1 at the origin of the graph of function $z = f(x, y)$, where

$$f(x, y) = \begin{cases} 0 & \text{for } x \leqslant 0, \\ x^2 & \text{for } x > 0 \end{cases}$$

(it appears at conjugacy points of a developable surface and a part of the plane; see point D in Fig. 2);

(4) the germ at the origin of the graph of function $z = f(x, y)$, where

$$f(x, y) = \begin{cases} x^2 & \text{for } y \leqslant x,\ x \geqslant 0, \\ y^2 & \text{for } y \geqslant 0,\ y \geqslant x, \\ 0 & \text{for } y \leqslant 0,\ x \leqslant 0; \end{cases}$$

(5) the germ at the origin of the graph of function $z = f(x, y)$, where

$$f(x, y) = \begin{cases} 0 & \text{for } y \leqslant 0,\ x \leqslant 0, \\ x^2 & \text{for } y \leqslant -x,\ x \geqslant 0, \\ y^2 & \text{for } y \geqslant 0,\ y \leqslant -x, \\ \frac{1}{2}(x^2 + y^2) - y - x & \text{for } x + y \geqslant 0 \end{cases}$$

(germs (4) and (5) correspond to the vertices B and A of planar triangles which appear in an essential way on the boundary of the convex hull);

(6) the germ at the origin of a truncated swallowtail (see Fig. 4)

$$f(x, y) = \min_{z \in \mathbb{R}} \{z^4 + xz^2 + yz\}.$$

More essential singularities occur in generic families of surfaces depending on three parameters. However, only some of them correspond to the boundary Σ of the transitivity zone.

Theorem 2 (Main Theorem). *For a generic family of curves* $r_m\colon S^1 \to \mathbb{R}^3$ *depending on a three-dimensional parameter* $m = (x, y, z) \in \mathbb{R}^3$, *the local singularities of the boundary* Σ *of the transitivity set are as follows* (*up to diffeomorphism of* $\mathbb{R}^3$):

(1) *the germs of the surfaces* (*of class* C^1 *or* C^∞) *of the convex hulls of typical surfaces in* $\mathbb{R}^3$ *listed above*;

(2.1) *the germ of the surface of a dihedral angle at the edge* (*see Fig.* 5),

(2.2) *the germ of the lateral surface of an n-gonal pyramid at its vertex, for* $n = 3, 4, 5$ (*for* $n = 5$ *one face cannot be straightened*, *see Figs.* 6–8);

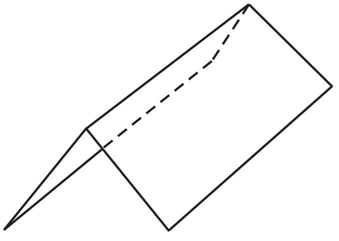

Figure 5

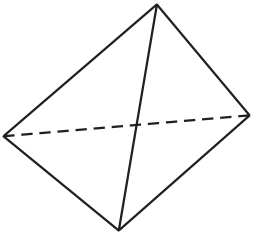

Figure 6

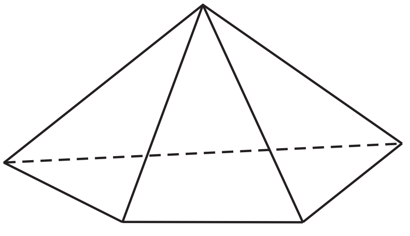

Figure 7

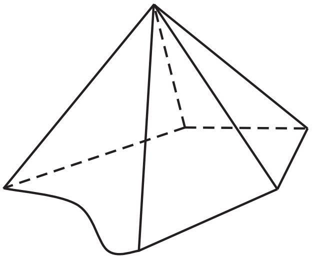

Figure 8

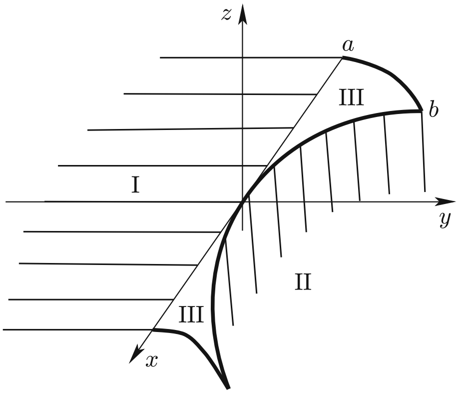

Figure 9

(3) *the germ at the origin of the union of three surfaces with boundary* (*see Fig.* 9), *given by the conditions*

$$z = 0, \qquad y \leqslant 0;$$
$$y = x^2, \qquad z \leqslant -4x^2;$$
$$z = -a^2, \qquad y = \frac{1}{4}z + ax, \quad a \in \mathbb{R}.$$

The first two surfaces (I and II) are smooth and transverse to each other (if we discard the inequalities). Their edges are smooth curves which are tangent at a single common point, the origin. The third surface (III) is part of the Whitney umbrella bounded by the edge curves of the first two surfaces and tangent to the first two surfaces at points of the edge curves (see Fig. 9).

§ 3. Proof of the main theorem

3.1. Auxiliary constructions and results. Here we use constructions similar to those used in the proof of the classification theorem for surfaces, see [1], Theorem 2. In the case of spatial curves these constructions are as follows.

Let q_1, q_2, q_3 be the coordinates in $\mathbb{R}^3$, and let $\widehat{\mathbb{R}^3} = \{(n_1, n_2, w)\}$ be the affine chart of the dual space consisting of cooriented planes in $\mathbb{R}^3$ which are not parallel to the q_3 axis. Then the oriented normal to such a plane has components $(n_1, n_2, 1)$, and w is the coordinate of the intersection point of the plane with the q_3 axis.

A *supporting plane* of a curve $I = \{r(t)\}$ is a co-oriented plane whose open positive half-space does not contain points of I, and the plane itself contains points of I. In other words, among all planes with normal $(n_1, n_2, 1)$ intersecting the curve I, the supporting plane P has the maximal value of w.

For each supporting plane P, we denote by S_P the set of its common points with I, and we refer to this set as the *support* of the plane P. Observe that all points in the support of a supporting plane are points of tangency with the curve.

The following statement is well-known (see, for example, [3]).

Proposition 3. *The convex hull of a compact set is the union of the convex hulls of supports S_P for all supporting planes P.*

Proposition 4. *For a family of generic curves depending on s parameters, each support S_P consists of at most $3 + s$ points.*

Proof. We consider the space of multijets at n points $r_1(t), \dots, r_n(t)$ of a parametrized curve $r(t)$, that is, the direct product of n copies of the space of jets $J^k(\mathbb{R}, \mathbb{R}^3) \times \cdots \times J^k(\mathbb{R}, \mathbb{R}^3)$ (of sufficiently high order k) of maps from the line to the three-dimensional space.

If there exists a plane tangent to the curve at n points corresponding to the values $t_1, \dots, t_n$ of the parameter t, then the vectors $\dot{r}(t_1), \dots, \dot{r}(t_n), r_1 - r_n, \dots, r_{n-1} - r_n$ are coplanar, that is, the rank of the $3 \times (2n-1)$-matrix formed by these vectors is at most 2. It follows from the theorem on the product of coranks that the codimension of the subset F given by this condition in the space of multijets is equal to $2n - 3$.

A family of curves depending on s additional parameters defines a map from the set of collections of these parameter values and n values $t_1, \dots, t_n$ to the space of n-multijets. It follows from the transversality theorem that for a generic curve this map is transverse to the above defined subset F in the space of multijets. In particular, for $2n-3 > n+s$, each multijet of a typical curve does not intersect this subset. Therefore, the inequality $n \leqslant 3+s$ is necessary for a plane and a spatial curve to be tangent at n points.

We consider multigerms of the curve in a neighbourhood of the support S_0 of the base supporting plane, which we may assume to be given by the equation $w = 0$. In a domain containing the convex hull of the support of the base plane, all other supporting planes belong to the above described chart of nonvertical planes, and their supports belong to a neighbourhood of the support S_0.

The Legendre transform γ plays the pivotal role in the classification of singularities of convex hulls of submanifolds in an affine space (see [3]). The Legendre transform of a spatial curve is defined as follows.

We assign to a point Q of the curve $r(t)$ the set of germs at this point. This set is diffeomorphic to the circle S^1 consisting of all cooriented tangent planes to this curve. The germs form the Legendrian submanifold $L_r \approx S^1 \times S^1$ in the space of co-oriented contact elements $ST^*\mathbb{R}^3$. Consider the projection γ_r of this submanifold, obtained by forgetting the base point of a germ and assigning the corresponding cooriented plane to it. Then γ_r is a Legendrian map from the submanifold L_r to the dual space $\widehat{\mathbb{R}^3}$.

Note that a plane from $\widehat{\mathbb{R}^3}$ is a supporting plane if and only if it is tangent to r at one or several points (that is, it belongs to the image of γ_r) and the value of w (that is, the coordinate of the intersection point of the plane with the q_3-axis) is maximal among all the parallel tangent planes from the image of γ_r. This image (the wavefront) $\widehat{r} = \gamma_r(L_r)$ is referred to as the Legendre transform of the initial curve r. The subset $\widehat{r}$ consisting of supporting planes will be denoted by $\mathrm{Su}(r)$.

A point $Q \in \mathbb{R}^3$ corresponds to a plane $\widehat{Q}$ in the dual space, where $\widehat{Q}$ consists of all planes of $\mathbb{R}^3$ passing through Q.

A point Q belongs to the boundary of the convex hull $H(r)$ if and only if $\widehat{Q}$ is a supporting plane for $\mathrm{Su}(r)$: the open negative half-space $\widehat{Q}$ does not contain points of $\mathrm{Su}(r)$, while the plane itself contains such points.

Therefore, the condition that a point O belongs to the boundary of the convex hull can be formulated in terms of the dual space $\widehat{\mathbb{R}}^3 = \{(n_1, n_2, w)\}$. Denote by $\widehat{O}$ the plane in $\widehat{\mathbb{R}}^3$ given by the equation $w = 0$ and consisting of all planes in $\mathbb{R}^3$ passing through the point O.

Proposition 5. *A point O belongs to the surface $X \subset \mathbb{R}^3$ if and only if the dual surface $\widehat{X} = \gamma(L_X)$ is tangent to $\widehat{O}$.*

Remark 6. If X is a ruled surface, then $\widehat{X}$ is a curve tangent to the surface $\widehat{O}$, for each $O \in X$. If X is a plane, then this condition is equivalent to the condition that $\widehat{O}$ passes through the point $\widehat{X}$.

The proof of the proposition follows from the fact that the square of the Legendre transform of a generic hypersurface is the identity transformation.

Degeneration of curves depending on parameters has been studied in [5] and [6], among other papers. In particular, the following statement can be extracted from these works, although it can easily be proved directly.

Proposition 7. *For generic families of embedded spatial curves r_m depending on a three-dimensional parameter $m = (x, y, z)$, the germ at a point in a proper affine coordinate system (q_1, q_2, q_3) with origin at this point has one of the following forms*:

(1) $q_1 = t$, $q_2 = t^2 + \cdots$, $q_3 = t^3 + \cdots$: *a nondegerate point of codimension* 0 (*type* A_2);

(2) $q_1 = t$, $q_2 = t^2 + \cdots$, $q_3 = t^4 + \cdots$: *a simple flattening point of codimension* 1 (*type* A_3);

(3) $q_1 = t$, $q_2 = t^2 + \cdots$, $q_3 = t^k + \cdots$, *where* $k = 5, 6, 7$: *a point of multiple flattening of codimension* $k - 3$ (*type* A_{k-1}); *this is the first essential singularity for isolated points of individual curves in families depending on* $k - 4$ *parameters*;

(4) $q_1 = t$, $q_2 = t^3 + \cdots$, $q_3 = t^k + \cdots$, *where* $k = 5, 6$: *a point of codimension* $k - 2$; *this is the first essential singularity for isolated points of individual curves in families depending on* $k - 3$ *parameters*;

(5) $q_1 = t$, $q_2 = t^4 + \cdots$, $q_3 = t^5 + \cdots$: *a point of codimension* 4; *this essential singularity occurs in isolated points of a curve for isolated values of three parameters.*

Here by the codimension we mean the codimension of the corresponding class in the space of germs (jets) $j^N(1, 3)$ at a fixed point. The dots denote the terms of higher order in t.

A germ at a point Q of the tangent plane to the curve is referred to as *regular* if the curve has singularity of type A_2 or A_3 at this point and the plane is not osculating, that is, does not coincide with the coordinate plane (q_1, q_2) in the above coordinates.

Note that in a typical case the space of parameters may contain a subset of codimension 1 corresponding to curves r_m with self-intersection points.

Proposition 8. *For a generic family of curves, if O belongs to a supporting plane P, then*

(1) *the plane P is either regular at each point of the support S_P, or is osculating at a simple flattening point of type A_3*;

(2) *P does not contain self-intersection points of the curve.*

Proof. The statement is implied by the following observations.

1. An osculating plane at a regular point cannot be supporting: the curve lies on both sides of this plane.

2. Let O belong to an osculating plane P for a degenerate singularity of the curve of codimension greater than one. Assume that P is a supporting plane and its support consists of $l = 1, 2, 3$ points. The dimension of the convex hull of the support is at most $l - 1$. At least $3 - l + 1$ conditions need to be satisfied for the point O to belong to the convex hull of the support. It takes l conditions for l points to belong to the osculating plane. Finally, at least one condition is required for the point itself to belong to the curve. Therefore, we need strictly more than three independent conditions, which is impossible in the case of general position with three parameters.

3. The codimension of the case when a support plane contains a self-intersection point of the curve, the point O, and possibly other points of the curve is calculated in the same way as in the previous paragraph, and is therefore greater than 3.

Proposition 9. *Assume that a point Q of the curve is either nondegenerate or a simple flattening point of the convex hull of the curve. Then the supporting planes containing Q form a closed arc on the circle S^1 of all co-oriented tangent planes at Q. Interior points of this arc correspond to supporting planes whose support consists of the point Q only, and the boundary points $E_1(Q)$ and $E_2(Q)$ correspond either to supporting planes whose support consists of more than one point, or to osculating planes (in the case of simple flattening).*

Proof. Let W be the subset of the circle formed by the tangent planes at Q. It is obvious that the closure of W is connected. The boundary points of W correspond to bitangent planes or to osculating planes.

The transversality theorem together with Theorems 1, 2 and Propositions 3–5, 7–9 imply the following list of possible locations of the origin on the boundary of the convex hull of a typical family of curves.

An ordered pair of points $r(t_1)$ and $r(t_2)$ on the curve is referred to as *collinear* if the velocity $\dot r(t_1)$ at the first point is collinear with the vector $r(t_2) - r(t_1)$.

The support of a supporting plane is said to be *noncollinear* if no pair of its points is collinear.

Lemma 10. *For a generic family of curves depending on a three-dimensional parameter, the parameter value m belongs to the boundary of the transitivity set Σ in one of the following cases.*

1_0. *The point O is a nondegenerate point or a simple flattening point of the curve r_m. The supporting planes have noncollinear supports, each of which consists of two points of regular tangency (Fig. 10).*

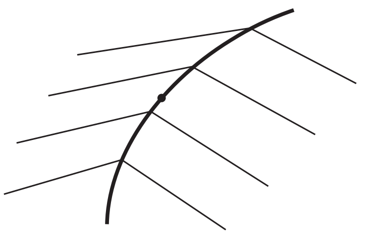

Figure 10

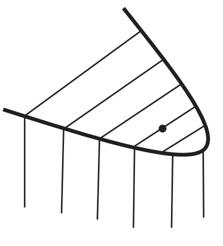

Figure 11

1_f. *The point O is a simple flattening point, the osculating plane is boundary and supporting, and its support consists of the point O only. The other boundary supporting plane has a regular tangency at one more point, which forms a noncollinear pair with O (Fig. 11).*

1_c. *The point O is a nondegenerate point of the curve, the supporting plane has support consisting of two points forming a collinear pair (Fig. 12).*

1_3a. *The point O is a vertex of a triangle which is the noncollinear support of a supporting plane, and the triangle is regular at all vertices. The arc of supporting planes at O degenerates into a point. The line containing the velocity vector at O intersects the interior of the triangle (Fig. 13).*

1_3b. *The point O is a vertex of a triangle which is the noncollinear support of a supporting plane. The plane has regular tangency with the curve at the vertices.*

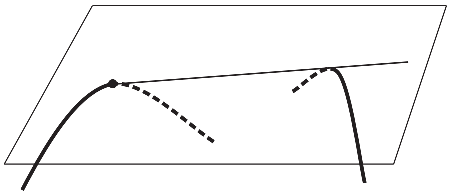

Figure 12

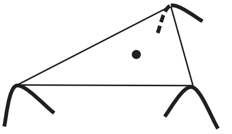

Figure 13

The line containing the velocity vector at O does not intersect the interior of the triangle. The other edge of the arc of supporting planes at O is a bitangent plane with regular tangency (*Fig.* 14).

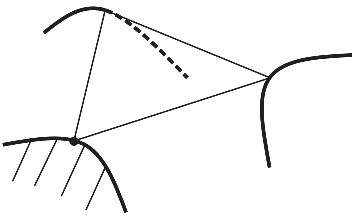

Figure 14

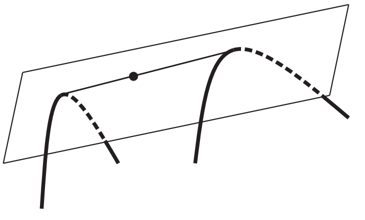

Figure 15

2_0. *The point O belongs to an open interval whose edges form the noncollinear support of a supporting plane with regular tangency at each of the edge points* (*Fig.* 15).

2_c. *The point O belongs to an open interval whose edges form the collinear support of a supporting plane with regular tangency at each of the edge points* (*Fig.* 16).

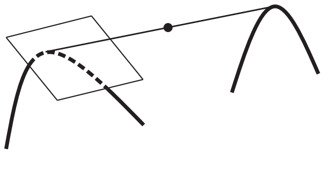

Figure 16

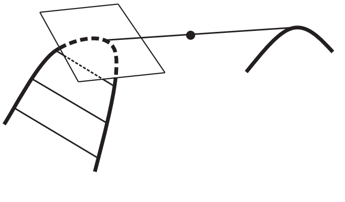

Figure 17

2_f. *The point O belongs to an open interval whose edges form the noncollinear support of a supporting plane with regular tangency at one of the edge points. The second edge point is a simple flattening point, and the osculating plane at this point coincides with the supporting plane* (*Fig.* 17).

3_0. *The point O belongs to the interior of a triangle whose vertices form the support of a supporting plane. The tangency at all vertices is regular, and the velocity vectors are not collinear with the edges of the triangle.*

3_c. *The point O belongs to the interior of a triangle whose vertices form the support of a supporting plane. The tangency at all vertices is regular, and the velocity vector at some vertex is collinear with an adjacent edge* (*Fig.* 18).

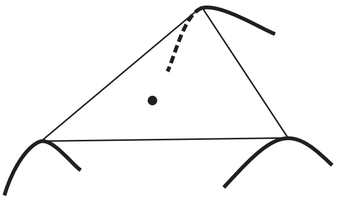

Figure 18

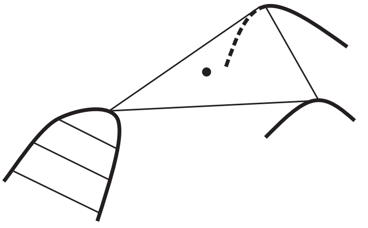

Figure 19

3_f. *The point O belongs to the interior of a triangle whose vertices form the support of a supporting plane. The tangency at two vertices is regular, and the third vertex is a simple flattening point whose osculating plane coincides with the supporting plane* (*Fig.* 19).

3_s. *The point O belongs to an edge of a triangle whose vertices form the support of a supporting plane. The tangency at all vertices is regular, and the velocity vectors are not collinear with the edges of the triangle* (*Fig.* 20).

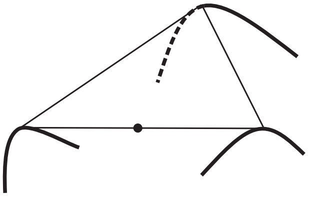

Figure 20

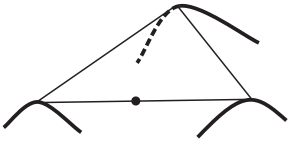

Figure 21

3_{sc}. *The point O belongs to an edge of a triangle whose vertices form the support of a supporting plane. The tangency at all vertices is regular, and the velocity vector at some vertex is collinear with an adjacent edge* (*Fig.* 21).

3_w. *Three points on a line form the support of a supporting plane. The tangency at all these points is regular, and the velocity vectors are not collinear with the line. The point O belongs to one of the segments formed by these three points* (*Fig.* 22).

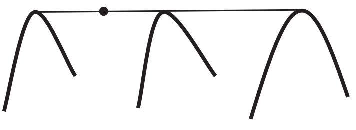

Figure 22

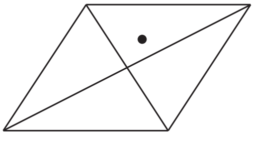

Figure 23

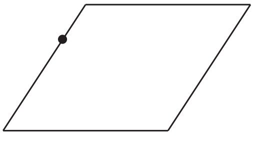

Figure 24

4_0. *The point O belongs to the interior of a quadrangle whose vertices form the support of a supporting plane. The tangency at all points is regular. The point O does not lie on the diagonals or edges of the quadrangle* (*Fig.* 23).

4_s. *The point O belongs to an edge of a quadrangle described in the previous case, so that all four vertices belong to one of the closed halfplanes with boundary containing the above edge* (*Fig.* 24).

4_{sb}. *The point O lies on a diagonal or on an edge of the quadrangle described in Case 4_0, so that the vertices of the quadrangle lie on different sides of the line containing the diagonal or edge* (*Fig.* 25).

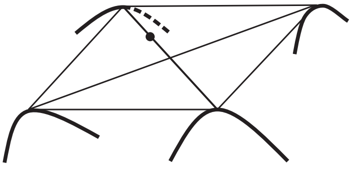

Figure 25

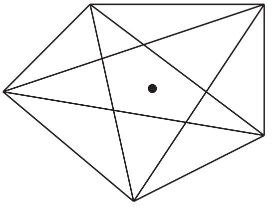

Figure 26

5_0. *The point O belongs to the interior of a pentagon with vertices $a_1, \dots, a_5$ and does not lie on the diagonals or edges of the pentagon. The points $a_i \in \Gamma$ form the support of a supporting plane P which has regular tangency with the curve at all points* (*Fig.* 26).

3.2. Proof of Theorem 2. We consider the germ of the boundary of the transitivity set at a base point m_0. The multigerm of the Legendrian map $(\gamma_{r_{m_0}}, P)$ is stable on its support S_P and, furthermore, the germ of the convex hull is stable in the vicinity of zero.

The convex hull is stable in the cases

$$1_0,\ 1_f,\ 1_c,\ 1_3a,\ 1_3b,\ 2_0,\ 3_0,\ 3_s,$$

and also in the cases

$$2_c,\ 3_c,\ 3_{cs}.$$

In cases 1_0 and 1_f the convex hull in the vicinity of zero consists of two transverse smooth surfaces with common boundary. In case 1_c the convex hull in the vicinity of zero is diffeomorphic to a truncated swallowtail. In cases 2_0, 3_0, 2_c and 3_c the convex hull is the germ at zero of a smooth surface which remains smooth for close values of parameters. In cases 3_s and 3_{sc} the germ at zero of the convex hull is also stable and consists of two smooth surfaces which are tangent at their common boundary, and therefore their union is a smooth surface of class C^1. Finally, in cases 1_3a and 1_3b the convex hull has type (4) or (5) from the list of normal forms of convex hulls of individual curves, and therefore the convex hull is also stable under the change of parameters.

Note that the collinearity of a pair of points a, b from the support of a supporting plane affects the stability of the convex hull at the point a only. Indeed, the Legendre transforms of germs of the curve at the points a and b are smooth surfaces $\widehat{a}$ and $\widehat{b}$ which intersect at the line corresponding to the pairs of points of tangency of bitangent planes with the germs of the curve. This line has a simple flattening point, and the osculating plane is dual to the point a. If for the initial value of parameter m_0 the origin O does not coincide with the point a, then for close values of the parameter the surfaces $\widehat{a}$ and $\widehat{b}$ are in general position with the surface $\widehat{O}$.

Thus, in all these cases, for close values of parameter m the boundary of the convex hull is diffeomorphic to the boundary $H(r_{m_0})$ consisting of the closures of domains on smooth surfaces.

We consider the union of germs $G_m = H(r_m) \times \{m\}$ in the direct product $\mathbb{R}^3 \times M$ of the phase space and the space of parameters. This union is a five-dimensional stratified set which is a topological manifold. The strata G_m are diffeomorphic to products of the strata of the convex hull $H(r_{m_0})$ (for $m = m_0$) and the germs of the space of parameters. In all cases except 1_c the closures of strata are smooth manifolds with boundary. In the case 1_c one stratum is the product of a truncated swallowtail and $\mathbb{R}^3$.

In general position, the three-dimensional submanifold $M_0 = 0 \times \{m\}$ of the product above is transverse to the submanifold G_m. Since G_m is a topological (Lipschitz) submanifold, the set of maps transverse to it is connected. By the Legendrian stability, the intersection $M_0 \cap G_m$ is diffeomorphic to the intersection of G_m with any transverse three-dimensional submanifold. In particular, this intersection is diffeomorphic to the section $m = m_0$, that is, to the initial boundary of the convex hull. Therefore, in all the cases when O belongs to the stable multigerm, the boundary Σ of the transitivity set coincides with the set $M_0 \cap G_m$ and is diffeomorphic to this germ. Thus, all the above cases have normal forms listed in part (1) of Theorem 2.

Now we consider unstable configurations.

As we have seen above, in the cases of collinear supports the instability of the convex hull may occur only at a point of the support from a collinear pair. In the dual space, the set of tangent planes to the germs of curves at the points of a collinear pair consists of smooth surfaces with transverse intersection. The only special feature of this case is that the line of intersection of the two sets of planes has an A_3-flattening at some point. In the case of general position the point O coincides with such a point in the stable case 1_c only.

Therefore, it remains to consider cases 4_0, 4_s, 4_{sb}, 5_0, 3_w, 3_f and 2_f.

We start with case 4_0, when the point O belongs to the interior of a quadrangle and does not lie on its diagonals or edges.

One of the possible locations of the support for $m = m_0$ is shown in Fig. 27. The point O lies in the interior of triangles ABC and BCD. Note that for any other location of the support there are two triangles with a common edge containing the point O.

We consider the image of the Legendre transform of all tangent planes at points of the germs of curves close to the points of the support $ABCD$. We obtain four germs of ruled smooth surfaces l_A, l_B, l_C and l_D corresponding to the vertices of the quadrangle. These surfaces intersect at a common point for $m = m_0$ and are in general position to each other. For parameter values close to m_0, the point O may belong to the convex hull only if it belongs to the interior of at least one of the triangles $\widetilde{A}\widetilde{B}\widetilde{C}$ and $\widetilde{B}\widetilde{C}\widetilde{D}$. These triangles are the supports of supporting planes and are close to the corresponding triangles ABC and BCD. Hence, for all close values of the parameter, if O belongs to the convex hull, then the surface $\widehat{O}$ contains one of the intersection points $\widehat{Q}_1$ and $\widehat{Q}_2$ of the corresponding surfaces l_A, l_B, l_C and l_B, l_C, l_D, provided that this point corresponds to a supporting hyperplane.

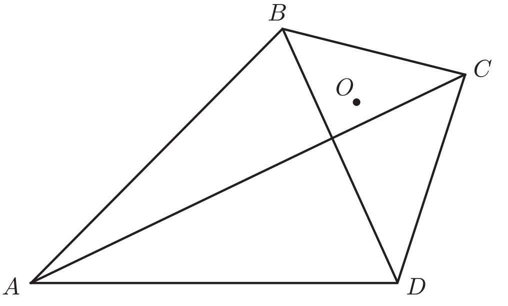

Figure 27

The distances from these points to the surface $\widehat{O}$ are two independent functions of the parameters f_1 and f_2. Since the points $\widehat{Q}_1$ and $\widehat{Q}_2$ lie on the curve l_{BC} of intersection of the planes l_B and l_C, the supporting plane is the plane with the greater distance.

Thus, the boundary Σ of the transitivity set is specified by the conditions $f_1 = 0$, $f_2 \leqslant 0$ and $f_2 = 0$, $f_1 \leqslant 0$. Taking the functions f_1 and f_2 as the coordinates in the space of parameters, we obtain the normal form given by a dihedral angle.

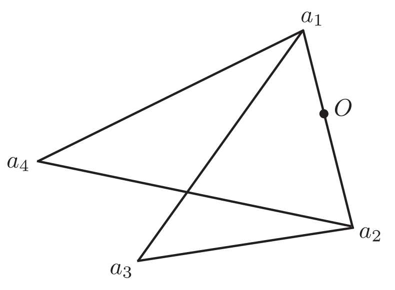

Figure 28

Now we consider case 4_s. Assume that for $m = m_0$ the point O lies on the edge a_1a_2 of a quadrangle $a_1a_2a_3a_4$ (Fig. 28). In the vicinity of O and for those m close to m_0, the boundary of $H(r_m)$ either belongs to one of the triangles $\widetilde{a_1}\widetilde{a_2}\widetilde{a_3}$ and $\widetilde{a_1}\widetilde{a_2}\widetilde{a_4}$ which are the supports (close to the corresponding triangles for $m = m_0$) of 3-tangent planes denoted by P_3 and P_4, or belongs to the ruled surface formed by the segments $\widetilde{a_1}\widetilde{a_2}$ lying in the bitangent supporting planes. For a fixed m, these bitangent planes form a smooth curve $P_t(m) \in \widehat{\mathbb{R}}^3$, which we parametrize by $t \in \mathbb{R}$.

Therefore, in case 4_s the set of supporting planes $\mathrm{Su}(r_m)$ intersecting with a neighbourhood of O is a part of the curve $P_t(m)$ corresponding to the interval $t \geqslant \max\{t_3, t_4\}$, where t_3 and t_4 are the parameter values corresponding to the points P_3 and P_4 on the curve. In the extended phase space $\widehat{\mathbb{R}}^3 \times M$ the curve $\bigcup_m P_t(m) \times \{m\}$ sweeps out a four-dimensional surface parametrized by t and m; for $m = m_0$ the curve $P_t(m_0)$ is tangent to the plane $\widehat{O}$.

Let $\theta\colon \widehat{\mathbb{R}}^3 \times M \to \widehat{\mathbb{R}}^3 \times M$ be a family of diffeomorphisms of $\mathbb{R}^3$ fibred over the parameter space, that is, $\pi \circ \theta = \overline{\theta} \circ \pi$, where $\overline{\theta}\colon M \to M$ is a diffeomorphism and $\pi\colon \mathbb{R}^3 \times M \to M$ is the projection onto the second factor. Assume that θ preserves the hypersurface $\widehat{O}$ and takes the set of supporting planes $\mathrm{Su}(r_m) \times \{m\}$ of one family r_m to the set $\mathrm{Su}(r'_m) \times \{m\}$ of another family Γ_m. Then the diffeomorphism $\overline{\theta}$ maps the corresponding sets Σ and Σ' to each other.

In our case, in general position there is a diffeomorphism $\theta\colon \widehat{\mathbb{R}}^3 \times M \to \widehat{\mathbb{R}}^3 \times M$ with the above properties which takes the curve $P_t(m)$ to the curve $w = t^2 - z$, $n_1 = t, n_2 = 0$, and the points P_3 and P_4 to the points $n_1 = x$, $n_2 = 0$, $w = x^2 - z$ and $n_1 = y$, $n_2 = 0$, $w = y^2 - z$, respectively. Indeed, in the case of general position the n_1-coordinates of 3-tangent planes have nondegenerate linear part in the variables x, y, z.

Therefore, the set Σ is diffeomorphic to the set specified by the following conditions: $z = 0$ for $x \leqslant 0$ and $y \leqslant 0$ (the interior points of the half-curve are tangent to $\widehat{O}$); or $z = x^2$ for $x \geqslant y$, $x \geqslant 0$; or $z = y^2$ for $y \geqslant x$, $y \geqslant 0$ (one of the points P_3, P_4 belongs to $\widehat{O}$). The resulting normal form coincides with the germ of the graph of function (4) from the list of normal forms of convex hulls of curves.

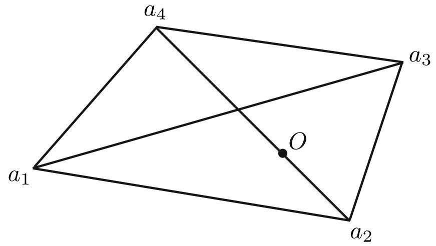

Figure 29

Now we consider case 4_{sb}. Assume that for $m = m_0$ the point O lies on the diagonal a_2a_4 of the support quadrangle $a_1a_2a_3a_4$ of the supporting plane (Fig. 29). For those m close to m_0 the boundary $H(r_m)$ of the convex hull in the vicinity of O is either the ruled envelope of bitangent planes $P_t(m)$ with supports $\widetilde{a}_2$, $\widetilde{a}_4$ close to the points a_2 and a_4, respectively, or one of the three 3-tangent planes: P_1 with support $\widetilde{a}_1\widetilde{a}_2\widetilde{a}_4$, P_2 with support $\widetilde{a}_2\widetilde{a}_3\widetilde{a}_4$ or P_3 with support $\widetilde{a}_1\widetilde{a}_2\widetilde{a}_3$.

Similarly to case 4_s, the set of supporting planes is either a segment of the curve $P_t(m)$ between the points P_1 and P_2 (if $t(P_1) \geqslant t(P_2)$), or, if the order of points on the curve is the opposite, then P_3 is a supporting plane. A diffeomorphism θ preserving $\widehat{O}$ takes the curve $P_t(m)$ to the form $w = t^2 - z$, $n_1 = t$, $n_2 = 0$, the point P_1 to the form

$$P_1 = \{n_1 = x,\ n_2 = 0,\ w = x^2 - z\},$$

the point P_2 to the form

$$P_2 = \{n_1 = -y,\ n_2 = 0,\ w = y^2 - z\}$$

and the point P_3 to a point on the hyperplane

$$w = \frac{x^2 + y^2}{2} - x - y - z.$$

Using the fact that P_3 becomes a supporting plane for $x+y \geqslant 0$, we obtain that the above normal forms define the germ of function (5) from the list of normal forms of convex hulls of curves.

The existence of such a diffeomorphism can be proved as follows.

Consider a germ of the surface γ_{r_m} at the point $\widetilde{a}_3$. We shall denote this surface by l_3; it contains all points P_i and curves described above. Choose local coordinates u, v on l_3 in such a way that the intersection of $\widehat{O}$ with l_3 is given by the equation $v = 0$. In general position, the intersection curves of l_3 with the other surfaces l_i $(i = 1, 2, 4)$ dual to the germs of the curve at the other points $\widetilde{a}_i$ are given by the equations $v = g_i(u, x, y, z)$, respectively. If we set the parameters x, y, z to zero, then the functions g_1 and g_2 have a simple zero at $u = 0$, while the function g_4 has a double zero at $u = 0$, which corresponds to a simple tangency of the line $P_t(0)$ with the plane $\widehat{O}$.

We observe that the subset Σ of the parameter space is a subset of the bifurcation diagram of zeros (that is, it is the set of parameters for which the function has a zero critical value) for the family of functions $V(u, x, y, z) = g_1 g_2 g_4$ on u with parameters x, y, z, and this family is of special type as the product of three factors. The theory of such (and more complex) composite families of functions has been developed in [7] and [8]. In particular, one consequence of this theory is as follows.

Proposition 11. *Each family V with the above described properties and with generic functions g_i can be taken by a contact equivalence fibred over the parameter space to a versal family of the form*

$$V^* = (u - x)(y - u)(u^2 - z)$$

in such a way that the sign of nonzero values of functions from V is preserved.

This proposition immediately implies the normal form of the boundary of the transitivity set in case 4_{sb}.

In case 5_0 we take the triangle $\widetilde{a_1}\widetilde{a_2}\widetilde{a_3}$ into a standard position by a family of affine transformations smoothly depending on parameter m. The boundary $H(r_m)$ of the convex hull in a neighbourhood of the q_3-axis is determined by one of several 3-tangent planes with supports from the neighbourhoods of $a_1, \dots, a_5$; these points are the vertices of the triangles which for $m = m_0$ contain the point O. It can be either three such triangles (domain I), or four (domain II), or five (domain III), see Fig. 30. In the way similar to the previous arguments, the 3-tangent plane which contains zero becomes a supporting plane and this defines the surface of a polyhedral angle in the parameter space. Note that one of the faces of a 5-hedral angle in a three-dimensional space can not be straightened in a general position (see also [1]).

In case 3_w, assume that for $m = m_0$ the origin belongs to the segment AB inside the segment AC, which is the convex hull of the support consisting of three collinear points $\{A, B, C\}$ on a supporting plane P_0 tangent to the curve in a regular way

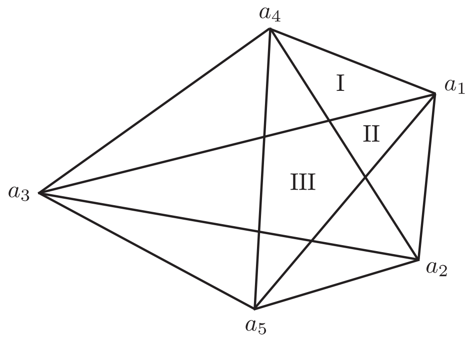

Figure 30

(the point B lies between A and C). The dual surfaces l_A, l_B, l_C of the germs r_m in a neighbourhood of $\widehat{P_0}$ are smooth (ruled) pairwise transverse surfaces. However, since the points A, O, B, C are collinear for $m = m_0$, the line ρ_1 of intersection of l_A with l_B is tangent to the line ρ_2 of intersection of l_A with l_C, and it is also tangent to the plane $\widehat{O}$ at the point $\widehat{P_0}$.

For an arbitrary m close to m_0, the boundary of the convex hull in the vicinity of zero is one of the ruled surfaces dual to the lines ρ_1 and ρ_2 lying on the surface l_A. Choose local coordinates u, v on this surface in such a way that the intersection of the plane $\widehat{O}$ with l_A is given by the equation $v = 0$. Then the curves $\rho_{1,2}$ are given by the equations $v = f_i(u, x, y, z)$, $i = 1, 2$, respectively, and for zero values of the parameters x, y, z the functions f_i have a nondegenerate (Morse) minimum, which is zero for $u = 0$.

As in case 4_{sb}, the boundary Σ of the transitivity zone consists of those values x, y, z for which the product $V = f_1(u, x, y, z)f_2(u, x, y, z)$ has a zero critical value which coincides with $\min_u \max_{i=1,2} f_i$.

The stability theory of complex functions (see [7], [8]) implies the following.

Proposition 12. *In general position, a contact equivalence fibred over the parameter space and preserving the sign of nonzero values takes the family V of functions f_1, f_2 to the form*

$$V^* = (u^2 - z)(4u^2 - 4xu + y).$$

Clearly, the contact equivalence described in the proposition takes the bifurcation diagrams of zeros of the families to each other, and it also takes to each other their subsets corresponding to Σ.

We can describe the set Σ in terms of the normal form V^*. First, the set Σ consists of those values of the parameters x, y, z for which the vertex of one of the corresponding parabolas belongs to the axis $v = 0$, provided that the vertex of the other parabola is below this axis. Second, the set Σ contains those parameter values for which the vertices of the parabolas do not belong to the graph of the auxiliary function $\max_i f_i(u)$ (the least value of this function is achieved at the point of intersection of the parabolas and is equal to zero). These conditions for V^* define exactly normal form (3) of Theorem 2.

In the case 2_f the boundary $H(r_m)$ for those m close to m_0 in the vicinity of O is either a ruled surface consisting of the support segments $[\widetilde{a}_1, \widetilde{a}_2]$ of the bitangent planes close to the support a_1, a_2 (this surface has an osculating supporting plane containing O for $m = m_0$), or a 3-tangent plane $\widetilde{a}_1$, $\widetilde{a}_1'$, $\widetilde{a}_2$ with points $\widetilde{a}_1$ and $\widetilde{a}_1'$ converging to a_1 as $m \to m_0$. For each m, the set $\mathrm{Su}(r_m)$ of supporting planes is the line of intersection of a truncated swallowtail $l_1 = (\gamma_{r_m}, \widetilde{a}_1)$ (the Legendre transform of the germ r_m in a neighbourhood of a_1) with a smooth surface $l_2 = (\gamma_{r_m}, \widetilde{a}_2)$. By using a diffeomorphism θ of the above type and vector fields tangent to the swallowtail, we can take our family of curves to the standard form

$$l_1 = \{(n_1, n_2, w) \mid \exists\, t : w = \min(t^4 + n_1 t^2 + n_2 t)\}, \qquad l_2 = \{n_1 = x\}.$$

In the case of general position such a diffeomorphism takes the surface $\widehat{O}$ to a surface of the form $w = z + y n_2 + \varphi(x, y, z, n_1, n_2)$. Here φ has the second order of smallness in the variables x, y, z, since $\widehat{O}$ is tangent to the section $n_1 = 0$ of the normalized truncated swallowtail for $x = y = z = 0$.

This normal form defines the set Σ consisting of those x, y, z for which the family $W = t^4 + xt^2 + n_2 t - z - y n_2 - \varphi$ of functions in t, n_2 has a critical point with zero critical value which is a minimum in t. In other words, the pair (y, z) belongs to the set of supporting lines of the intersection curves of the minimal part of the swallowtail,

$$w = \min_{t, n_2} \{t^4 + xt^2 + b(x, y, n_2)t + c(x, y, n_2)\}$$

with the plane $x = \mathrm{const}$, for certain smooth functions b, c.

By taking the family W to the normal form by contact equivalences (preserving the sign of nonzero values) we obtain that the surface Σ is diffeomorphic to the germ at the origin of the graph of function $z = \min_y(y^4 + xy^2)$. This is a C^1-smooth conjugation of two surfaces with a common boundary (type 2) from the list of normal forms of convex hulls of typical curves.

Finally, we consider the last unstable case 3_f. In the vicinity of O for those m close to m_0 the boundary $H(r_m)$ is given by the 3-tangent plane $\widetilde{a}_1 \widetilde{a}_2 \widetilde{a}_3$, which is close to the supporting osculating plane $a_1 a_2 a_3$ for $m = m_0$. Therefore, the set $\mathrm{Su}(r_m)$ is the intersection of the surface of a truncated swallowtail $l_1 = (\mathrm{Su}(r_m), a_1)$ with the smooth surfaces $l_2 = (\gamma(r_m), a_2)$ and $l_3 = (\gamma(r_m), a_3)$. These smooth surfaces l_2, l_3 and $\widehat{O}$ are pairwise transverse for $m = m_0$. They remain transverse for close values of m. In general position, the smooth map ρ taking the parameter m to the intersection point of these three surfaces is nondegenerate. Therefore, the set Σ consisting of those points m for which $\rho(m)$ belongs to the truncated swallowtail l_1 is diffeomorphic to the latter set. Thus, we obtain normal form (4) from the list of normal forms of singularities of convex hulls of spatial curves.

This finishes the proof of Theorem 2.

3.3. Proof of Theorem 1. The proof of Theorem 1 is now an easy exercise, which consists in applying the Legendre transform to planar curves and checking the following facts.

The support of a supporting line of a curve from a generic two-parameter family contains at most three points, provided that the origin belongs to the convex hull of this support. If the origin belongs to a one- or two-point support, then the convex hull is stable in the vicinity of zero in a general position.

In the remaining unstable case of a three-point support the Legendre transform of the multigerm of the curve consists of three pairwise transverse curves, which depend on common parameters. The location of zero inside the convex hull is determined by the competition between the two intersection points of these lines, and defines a singularity of type $y = |x|$ in the way similar to case 4_0.

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A. N. Kurbatskiĭ
Moscow School of Economics,
Moscow State University
E-mail: akurbatskiy@gmail.com

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